Topological Quantum Field Theory:
A Prosperous Link Between Physics and Mathematics

José M. F. Labastida
Universidade de Santiago
Santiago de Compostela, Spain

Abstract

Quantum Field Theory has played a fundamental role in our understanding of the behavior of elementary particles. In the eighties it was discovered that quantum field theory could also be a very useful tool to study some aspects of low-dimensional topology, and the concept of Topological Quantum Field Theory was introduced. The richness of quantum field theory encoded in its different methods of study has been applied to this new concept, and new unexpected results have been obtained. The introduction of Seiberg-Witten invariants and of their relation to Donaldson invariants on four-manifolds, as well as the construction of integral representations of Vassiliev invariants for knots and links on three-manifolds, are two of the most salient accomplishments of topological quantum field theory. These have been achieved by a combination of some of the perturbative and nonperturbative methods of quantum field theory. From these results there emerges a new picture for some sets of topological invariants in which these are classified in terms of universality classes.

1 Introduction

During the last decade we have witnessed the emergence of a remarkable new relation between physics and mathematics. The most advanced elements of theoretical physics have become tools to create new mathematics. This type of relation is unprecedented in this century. It is also different than the usual relations in previous centuries in which often new mathematics were created because they were needed to describe physical situations. In the present case there is no such a need: physical theories are now used because they are able to provide new insights in mathematics whose present relevance comes entirely from the mathematical side. The field of theoretical physics which takes part in this relation is quantum field theory, and the special quantum field theories which are involved are called topological quantum field theories (TQFTs).

Quantum field theories are physical theories which are both quantum and relativistic. This means that they implement consistently two of the main physical principles discovered in this century: quantum mechanics and special relativity. These theories are therefore used to describe physical situations in which quantum and relativistic effects are important. They have been very successful in the description of the behavior of elementary particles at high energies. The Standard Model, which is based in quantum field theory, has been confronted with experiments to a high degree of accuracy. However, quantum field theory and the Standard Model itself have many problems and leave many questions unanswered. For example, quantum field theory is based on functional integrals, which are in general not well defined, and the Standard Model leaves aside gravity, one of the four fundamental interactions.

From a theoretical point of view the situation is rather unsatisfactory. This has led theoretical physicists to develop a variety of methods to study quantum field theory, and to consider a new kind of quantum theory which could accommodate gravity consistently. The methods are classified mainly in two types: perturbative and non-perturbative. On the other hand, with regard to the new kind of quantum theory, there exists at the moment a very promising theory, string theory, which certainly incorporates gravity and, furthermore, it might provide a unified theory involving all the fundamental interactions. The problem is that we do not know yet how to correctly formulate it.

A series of important events occurred in the eighties which made us turn into the new decade with a very promising tool to develop. In 1982, S. Donaldson discovered that the study of instantons, objects which appear in quantum field theories when they are analyzed from a non-perturbative point of view, provides very important information to study compact oriented smooth four-manifolds. Also in 1982, E. Witten, trying to unravel the structure of two-dimensional sigma models, generalized Morse theory to what is now known as Morse-Witten theory, an ancestor of TQFT. This theory was later rigorously reformulated by A. Floer who applied similar ideas to compact three-manifolds, constructing in this way new important objects from a topological point of view. In 1988, E. Witten, inspired in part by the work by Floer, formulated the first TQFT which in fact contains the topological invariants first studied by Donaldson at the beginning of the decade. The resulting TQFT is known as Donaldson-Witten theory.

In this brief history of the eighties there are two other important protagonists who played fundamental roles. One is M. Atiyah who soon was convinced that Donaldson theory could be formulated in terms of quantum field theory. His efforts to construct such a theory and to attract Witten to think on the problem were crucial. The second is string theory. String theory had a vertiginous development after 1985. Many theoretical physicists jumped in those days to heavily work on this theory. This development was strongly influenced by topological and geometrical ideas, creating a fruitful atmosphere for TQFT. A scenario where a quantum field theory of topological type could fit was found in 1987. It was then discovered that at high temperature strings could be described in terms of a theory with no degrees of freedom. The formulation of a theory with a feature like this, known then as new phase of gravity, was a goal whose achievement influenced Witten to construct his first TQFT in 1988. This first relation between string theory and TQFT did not have important consequences. However, it is very likely that string theory will provide a very useful tool to study the topology of low-dimensional manifolds and perhaps this will be the new breakthrough that we will witness in the second half of the present decade.

The eighties were completed by the formulation by Witten of two other fundamental TQFTs: topological sigma models for two-dimensional manifolds, which contain the Gromov invariants, and Chern-Simons gauge theory for three-manifolds which contains knot invariants as the Jones polynomial and its generalizations.

The present decade started with the work by Atiyah and Jeffrey on the formulation of TQFT using the Mathai-Quillen formalism. That work provided a general framework to understand the meaning of certain type of TQFTs from a mathematical point of view. However, it was not very useful to solve these theories. The first half of the present decade is characterized in fact by the opposite. The application of physical methods to certain class of TQFTs has led to their solution and to obtain a entirely new point of view from a mathematical perspective. The main physical concept which has been involved in this remarkable development is duality. Its use by Seiberg and Witten has originated a revolution in the program on four-manifolds started by Donaldson. They provided a framework which contains simpler but somehow equivalent topological invariants. These invariants are known as Seiberg-Witten invariants. It is very likely that this new framework will open new scopes not only in four dimensions but in other low-dimensional manifolds. Though these results are very astonishing, it is not unplausible that string theory is behind all this and that we have discovered only a small fraction of what will be found once string theory is understood.

In this talk, after introducing quantum field theory and TQFT I will describe, using the Aharonov-Bohm effect, why topology is relevant in quantum mechanics. This will allow us to get into Chern-Simons gauge theory and its knot invariants. Its various studies will permit us to understand the usefulness of both, perturbative and non-perturbative methods, and will allow us to discuss Vassiliev invariants for knots. Then we will leave these theories and will start with supersymmetric gauge theories and the TQFTs which are derived from them. Duality properties of supersymmetric gauge theories will be then applied obtaining the new framework which contains Seiberg-Witten invariants. I will end describing some generalizations which induce the idea of universality classes of topological invariants which is in part already present in Chern-Simons gauge theory.

2 Quantum field theory and TQFT

As was already mentioned in the introduction, quantum field theory is a theory which reconciles quantum physics with special relativity providing a helpful framework to describe the behavior of elementary particles. We cannot go here into details but we can give a general picture on how this theory is used and what are the mathematics involved.

As in any other theory, in quantum field theory one begins considering a set of input data and then computes some quantities which are of interest because in principle they could be measured in laboratories. These quantities are the predictions of the theory. The standard experimental setting which is behind quantum field theory consists of a collision in which incoming and outgoing particles participate, the input data being the classical properties of these particles, their masses, their momenta, their spins, etc.. Given a specific situation, quantum field theory is the tool to be used to compute quantities which could be measured such as cross sections, decay rates, etc. These quantities are basically probabilities for a given event characterized by the input data to happen.

Once we have a picture of what is involved in quantum field theory let us describe the type of mathematics which one has to confront in doing the calculations needed to obtain the probability for an event to occur. The basic ingredient is the generalization to the case of fields of the Feynman path integral. In quantum field theory one first associates a field $\Phi_{m_i,p_i,s_i,\dots}(A)$ to each particle of the input data. This field contains the information which characterizes the state of particle i, namely, its mass, m_i, its momentum, p_i, its spin, s_i, etc., and is expressed in terms of the basic fields of the theory which are collectively denoted by A. The quantity which one computes and is associated to the probability for the event to happen is called vacuum expectation value of the product of fields $\Phi_{m_i,p_i,s_i,...}(A)$, $i = 1, \ldots, n$, and it is basically the average value of this product weighted by a function which contains the most fundamental ingredient of the theory: the action or integral over space-time of the lagrangian density. Vacuum expectation values are denoted by open brackets and have the following form,

$$\begin{aligned} & \langle \Phi_{m_1,p_1,s_1,...} \Phi_{m_2,p_2,s_2,...} \cdots \Phi_{m_n,p_n,s_n,...} \rangle \\ = \; & \frac{1}{Z} \int [DA] \, \Phi_{m_1,p_1,s_1,...}(A) \, \Phi_{m_2,p_2,s_2,...}(A) \cdots \Phi_{m_n,p_n,s_n,...}(A) \\ & \exp\big(iS(A)\big), \end{aligned} \tag{2.1}$$

where $[DA]$ denotes some integration measure over the space of configurations of the basic fields, $S(A)$ denotes the action, and Z is the partition function of the theory:

$$Z = \int [DA] \exp\big(iS(A)\big). \tag{2.2}$$

Out of the three ingredients of a quantum field theory, the one on which we have more control is on the action $S(A)$. The form of the action is in general very much constrained by the symmetries of the theory. For example, in the case of the Standard Model, the presence of a gauge symmetry based on the gauge group $SU(3) \times SU(2) \times U(1)$ severely constraints its form. The action in this case is known except for a small fraction of it and the part which is widely accepted has been tested experimentally to a high degree of accuracy. The fields $\Phi_{m_i,p_i,s_i,...}(A)$, $i = 1, \ldots, n$, are harder to control, specially in theories like quantum chromodynamics, which is part of the standard model, in which the property of confinement takes place. But the really unsurmountable problem is to define a measure for the functional integration involved in the computation of vacuum expectation values. It is not known in general how to do it. This has led theoretical physicists to develop a variety of methods to circunvent the problem. In fact, the richness of quantum field theory resides in the existence of this variety of methods which in practice turn out to be complementary since each of them provides partial information on the structure of the quantum field theory involved. As mentioned in the introduction, these methods usually fall into two categories: perturbative and non-perturbative. One of the main goals of this talk is to explain precisely how the application of these methods to TQFT has led to the recent successful results which have changed our way of looking at certain sets of invariants of low-dimensional manifolds.

It is now the turn of TQFT. These theories are special cases of quantum field theories. One of the properties which singularizes these theories is that now the space-time in which they are defined is a general smooth manifold and that the input data are not labels of particles but labels of topological or geometrical origin related to that manifold. These labels might be, for example, homology cycles, loops, etc. Another property which characterizes TQFTs is that their actions are such that the resulting vacuum expectation values do not depend on the metric on the manifold. The result of the computation of a vacuum expectation value in TQFT does not have an interpretation as a probability for an event to happen. These quantities turn out to be topological invariants. The reason for this is that they correspond to quantities which do not vary under deformations of the metric.

As in ordinary quantum field theory, the hard problem in TQFT is to define properly the functional integration measure. The problem of finding the equivalent of the fields $\Phi_{m_i,p_i,s_i,\ldots}(A)$ is much simpler in this case. Due to the problem with the measure one cannot think of the results so obtained with TQFT as rigorous from a mathematical point of view. Perturbative and non-perturbative methods are used to obtain those results and these methods contain a part based on the intuition that physicists had acquired through their work during many years trying to make sense of quantum field theory and confronting their results with experiments. The rigorous mathematical work that definitely describes the invariants predicted by TQFT is carried out using different methods. This work is certainly necessary and completes the formulation, making this new relation between physics and mathematics very fruitful. It is likely that in the future the arrow will turn backwards and physics will profit having at its disposal an elaborate theory on functional integration. This would be a very rewarding outcome of this relation.

3 Topology and quantum mechanics: the Aharanov-Bohm effect

The two branches of physics and mathematics which are particularly involved in TQFT are quantum mechanics and topology. At first sight, one would not anticipate a relation between the two. However, there is a simple qualitative argument to expect a link between them: both, topology and quantum mechanics, lead to discrete quantities out of continuous data. One could think for example of the Euler number for smooth manifolds in the case of topology, or the spectrum of the hydrogen atom in the case of quantum mechanics.

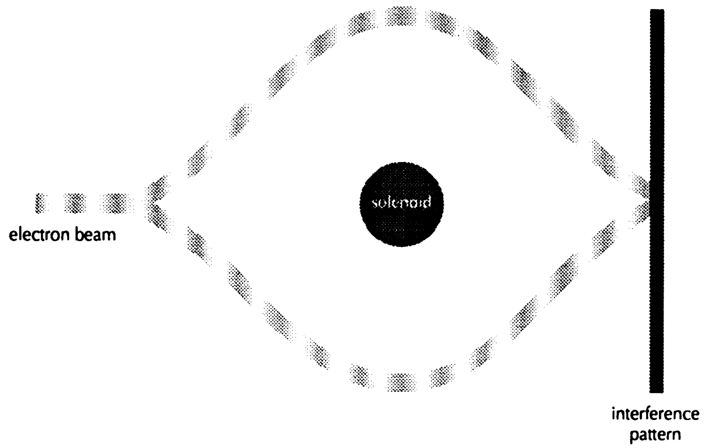


Figure 1: Sketch of the experiment proposed by Y. Aharanov and D. Bohm.

The deep relation between topology and quantum mechanics became manifest after the Aharanov-Bohm effect was understood in the late fifties. In 1959, Yakir Aharanov and David Bohm proposed an experiment which showed that the global properties of space-time were important in the description of quantum processes. The experiment was carried out in 1960 by R. G. Chambers and since then the physical process which takes place is known as the Aharanov-Bohm effect. In order to understand the role played by topology in this effect let us briefly describe the experimental setting in which it is observed and its theoretical explanation in terms of quantum mechanics.

The experimental arrangement consists of a very thin and long solenoid, which creates a magnetic field, and an electron beam which is split into two partial beams, each traveling along one side of the solenoid. The two partial beams are then recombined so that they interfere. A transversal section of the experimental situation is schematically depicted in Figure 1. For a thin and long enough solenoid the setting is such that the magnetic field vanishes along the paths travelled by the two partial beams. This means that at least classically one would expect that the interference pattern would be the same whether or not an electric current goes through the solenoid. This is not what is observed experimentally. Chambers found in 1960 that the interference pattern gets shifted when the current going through the solenoid is increased. This effect does not have an explanation classically. According to the classical equations of electrodynamics, Maxwell's equations, if the magnetic and the electric fields vanish, charged particles do not feel the interaction. In quantum mechanics, however, the interaction between electromagnetic fields and particles is described making use of the electromagnetic potential. If one could argue that the electromagnetic potential is different in each region travelled by the partial beams, and that such a difference depends on the current going through the solenoid, one could explain the shift in the interference pattern which is observed. This is in fact the way to understand the Aharanov-Bohm effect.

The first question which we must address to analyze the experiment is what is the value of the electromagnetic potential in each region. This does not have a unique response due to the existence of a gauge symmetry. The presence of a gauge symmetry implies that there are several descriptions in terms of electromagnetic potentials. Each description is related to the others by a gauge transformation. To choose one specific description is called to choose a gauge. One obvious question to ask is if in a situation in which the magnetic field is zero one could always choose a gauge in which the vector potential (part of the electromagnetic potential associated to the magnetic field) vanishes. If the answer were positive, one would not be able to explain the Aharanov-Bohm effect the way we intend to do. But if that were the case one would enter also in contradiction with Maxwell's equations. According to these equations, the integral of the vector potential $\mathbf{A}$ along the loop C pictured in Figure 1 should equal the magnetic flux Φ through the solenoid:

$$\oint_C \mathbf{A}d\mathbf{l} = \Phi. \tag{3.1}$$

When the current through the solenoid is turned on, the magnetic flux Φ is not zero and Maxwell's equations would be inconsistent for a null vector potential. If one thinks instead that $\mathbf{A}$ is pure gauge one is still in trouble, because then $\mathbf{A} = \nabla\phi$ for some scalar function ϕ, and then the left hand side of (3.1) would vanish. The puzzle is solved in the theory of electromagnetism allowing multivalued functions ϕ or, equivalently, vector potentials which are defined only locally. If ϕ were multivalued the left hand side of the last equation would not vanish in general. In such a situation the difference in the value of ϕ when one goes along the loop C should be just the magnetic flux. Notice that this picture does not lead to any singularity due to the fact that there is a region excluded: the region containing the solenoid. In other words, the space where the description is valid is not simply connected. A different but equivalent point of view consists of splitting space in regions which overlap (patches) and assume that the vector potential is different in each region while differing in the overlapping regions by gauge transformations. For a non-vanishing magnetic flux Φ two regions are enough to obtain a satisfactory description consistent with equation (3.1). Again, this framework does not lead to singularities due to the fact that space is not simply connected.

The mathematics behind the description based in a vector potential only defined locally is the theory of principal fiber bundles. The vector potential plays the role of a connection. Thus the mathematical description is intrinsically related to geometry and topology. In either description the vector potential is different in the region travelled by each partial beam and therefore, since in quantum mechanics charged particles couple to the vector potential, one expects a shift in the interference pattern as the flux Φ or, equivalently, the electric current through the solenoid, is increased. The detailed mathematical analysis leads precisely to the prediction of a shift in the interference pattern which is in full agreement with the one which is observed experimentally.

The Aharanov-Bohm effect was the starting point of a continuous presence of geometry and topology in quantum physics. The crucial point is that in quantum physics interactions are described by potential fields and these objects have a fundamental meaning from the point of view of geometry and topology. Since then many objects of geometrical or topological origin have played an important role. The most important case is non-abelian gauge theory, which is a generalization of electromagnetism in which the potential is a connection associated to a non-abelian group, in contrast to the case of electromagnetism in which it is abelian. The roots of many developments in theoretical physics during the last decades are based on objects of geometrical or topological origin. Examples of this are magnetic monopoles, solitons, instantons, strings, etc. The Standard Model itself is a non-abelian gauge theory whose gauge group is $SU(3) \times SU(2) \times U(1)$.

4 Chern-Simons gauge theory and link invariants

The effect described in the previous section revealed the importance of geometry and topology in quantum mechanics. In fact, the quantity that is computed integrating along the path C is a topological quantity. If one slightly deforms the path C, the value of the integral remain unchanged or, if one goes around the solenoid one more time one gets twice the magnetic flux. The path integral of the vector potential is proportional to the number of times that the path winds around the region of space which is excluded. This winding number is clearly topological.

To obtain more interesting topological quantities one can think of replacing the electromagnetic field of the Aharanov-Bohm effect by a non-abelian gauge field. In fact, this could lead to an interesting theory from a geometrical or topological point of view without the solenoid because non-abelian gauge theories, contrary to electromagnetism, are self-interacting. However, the situation is not so simple for two reasons: first, the path integral of equation (3.1) is not gauge invariant and one has to consider its gauge invariant generalization, the Wilson loop; second, for this quantity small deformations of the path C imply a change in its value.

The two problems plus the self-interaction property get nicely combined if one lowers the dimension of space-time and chooses a special action for the corresponding gauge theory: the Chern-Simons action. This action is based on a geometrical object known as the Chern-Simons form. The value of a Wilson loop remains invariant under deformations of the integration path which do not lead to crossings. Thus, in this theory, to each loop or set of loops embedded in three-dimensional space one gets a quantity which is invariant under small deformations which do not imply crossing lines. One seems to be dealing with topological quantities. Furthermore, these quantities are not trivial because Chern-Simons gauge theory is self-interacting. This theory possesses a contact interaction which modifies the value of the Wilson loop when lines cross to each other or to themselves. Indeed, the quantities associated to these sets of embedded loops in three-dimensional space are knot invariants.

A knot is a one-dimensional curve traced in three-dimensional space in such a way that it begins and ends at the same point and does not intersect itself. A link is a set of one-dimensional curves of the same type which do not intersect to each other. Knottedness and linkedness are not properties of the curves but of the way they are embedded in three space. Knots and links are specific of three dimensions, precisely the dimension for which Chern-Simons gauge theory exists. In Figure 2 some simple knots and links are shown: the first three are knots (or links of one single component) and the fourth is the simplest among two-component links: the Hopf link.

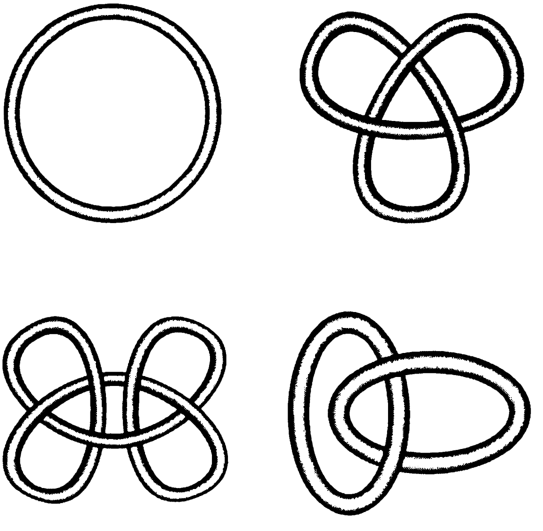


Figure 2: Some examples of knots and links: unknot, trefoil, squarea knot and Hopf link.

Interest in knot theory started in the 19th century when William Thompson (Lord Kelvin) proposed a model for atoms based on knots. Though this idea was soon discarded to describe atoms, it aroused interest in the problem of classifying knots. In 1900 Peter G. Tait published the first table of knots and links, and formulated a series of conjectures that in some cases waited eighty years for a proof. Since then knot theory has been a field of interest in mathematics. It has been very fruitful in its application to the study of the topology of three-manifolds.

One of the goals of knot theory is to classify knots and links. Two links (often in this paper knots will be treated as links of one component) are topologically equivalent if one can be obtained from the other by a continuous deformation, in other words, when no intersections occur in the deformation. Thinking of links as a series of knotted and linked strings with their loose ends attached, two links are equivalent if one can be deformed into the other without breaking any of the strings. In Figure 2 no pair of links contains two which are topologically equivalent. This statement, though it does not have a simple proof, seems plausible due to the simplicity of the links involved. However, for two complicated links it may be extremely difficult to decide whether they are topologically equivalent.

Mathematicians have developed techniques to discriminate between links. One of these techniques is based on the construction of link invariants or quantities associated to links which are invariant under continuous deformations. Two links having different link invariants are topologically inequivalent. However, two links having the same invariant might or might not be topologically equivalent. The more discrimination is achieved by a link invariant the better, but as yet there is not a complete classification of links.

In 1923 W. Alexander introduced a polynomial link invariant which had a good discrimination power as compared to previous invariants. However, it was soon realized that many topologically inequivalent links had the same Alexander polynomial. For example, knots which are not topologically equivalent to their mirror image knots (as the trefoil knot) have the same Alexander polynomial. Fundamental progress in knot theory was achieved by V. F. R. Jones in 1984 after the discovery of a new polynomial link invariant. This invariant is much more powerful than the Alexander polynomial; for example, in general, it distinguishes knots from their mirror images when they are not topologically equivalent. Nevertheless, soon it was discovered that there are non-equivalent knots with the same Jones polynomial. Invariants with more discrimination power were needed. After Jones' discovery, other polynomial invariants as the HOMFLY polynomial were constructed. Many of these new invariants, like the Jones polynomial itself, were formulated from mathematical structures whose study was in part motivated by statistical mechanics.

Chern-Simons gauge theory was formulated in 1988 providing an entirely new point of view in knot theory. This gauge theory is three-dimensional and so it provides an intrinsically three-dimensional formulation of polynomial link invariants. All previous formulations of these invariants were basically two-dimensional, defined on plane projections. This feature allows to obtain link invariants for arbitrary smooth oriented three-manifolds, and not only for flat space or for the three-sphere as was the case in previous formulations. In Chern-Simons gauge theory there exists a polynomial invariant for each representation of each simple Lie group. All previous polynomial invariants correspond to some specific choice of group and representation, or a special limit of some of them. It is not known yet if this huge amount of link invariants discriminates all topologically inequivalent links.

Chern-Simons gauge theory possesses the general problems of any quantum field theory, in particular, its integration measure is not well defined. However, being topological, it is simpler than the ordinary ones. Non-perturbative methods have been applied to this theory leading to its exact solution, at least for the case of simple three-manifolds. Chern-Simons gauge theory is one of the few quantum field theories whose exact solution is known. The solution consists in a series of rules which allow to compute vacuum expectation values of any product of Wilson loops. These rules are particularly simple for special cases. For example, for the case of the gauge group $SU(2)$ and Wilson loops in its fundamental representation the rule is shown in Figure 3. This rule has to be understood in the following way: project the link on a plane labeling overcrossings and undercrossings. Then, for three links which differ only in a part as depicted in Figure 3 the relation between the corresponding vacuum expectation values is:

$$\frac{1}{t}W_{L_+} - tW_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})W_{L_0}, \tag{4.1}$$

where t is a function of the coupling constant, $g = 1/\sqrt{k}$, of the theory:

$$t = \exp\left(\frac{2\pi i}{k+2}\right). \tag{4.2}$$

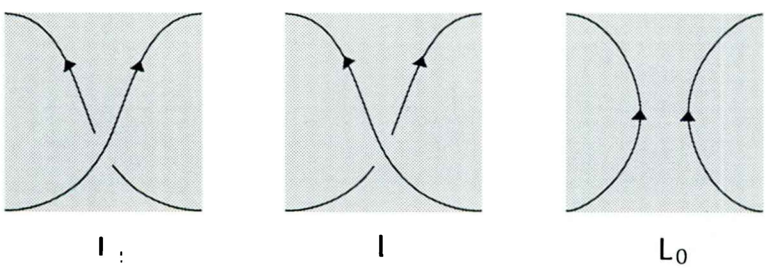


Figure 3: Skein rules for the Jones polynomial.

The rule so obtained (called skein rule) is precisely the rule which defines the Jones polynomial. The normalization is taken usually in such a way that for the unknot the polynomial invariant is 1. It is clear from this rule that the resulting invariant is a polynomial in $\sqrt{t}$ with positive and negative powers.

Chern-Simons gauge theory leads to a link invariant for each irreducible representation of each simple group. In general one does not find a skein rule as simple as the one for the Jones polynomial, but with the help of other non-perturbative methods one can complement the skein rule to design a calculation procedure. One important property of the solution found is that the vacuum expectation values are analytic in the coupling constant, $g = 1/\sqrt{k}$. This implies that the power series that results from a perturbative approach has to match the power series in $g = 1/\sqrt{k}$ streaming from the exact solution. When a situation like this occurs one says that there are not non-perturbative effects in the theory. But why worry about the power series if one knows the exact sum? There is an important reason for this. Perturbation theory provides path and space integral expressions for the coefficients of the power series expansion. If the whole series is a link invariant, each coefficient is also a link invariant since a continuous deformation of the link changes the expressions for the integrals of the coefficients but not the expansion parameters $g = 1/\sqrt{k}$.

Let us consider the trefoil and its Jones polynomial as an example. This polynomial is:

$$W_T = t + t^3 - t^4, \tag{4.3}$$

which, after expanding in powers of the coupling constant, results in:

$$W_T = 1 - 12\left(\frac{\pi i}{k}\right)^2 - 48\left(\frac{\pi i}{k}\right)^3 + \dots \tag{4.4}$$

In doing this calculation one removes first the shift by 2 of k in the denominator of the exponential in (4.2). This shift is controlled in perturbation theory by loop insertions related to finite renormalizations and can be ignored if one discards contributions from Feynman diagrams corresponding to those insertions. The integral expression which is provided by perturbation theory for the -12 appearing in the expansion (4.4) is the following:

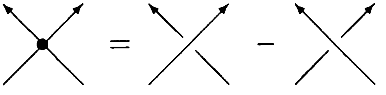
$$\begin{aligned} -12 \quad &= \frac{1}{2} - \frac{3}{4\pi^2}\oint_T dx_\mu \int^x dy_\nu \int^y dz_\rho \int^z dw_\tau \Delta^{\mu\rho}(x-z)\Delta^{\nu\tau}(y-w) \\ &+\frac{3}{16\pi^3}\oint_T dx_\mu \int^x dy_\nu \int^y dz_\rho \int d^3\omega \epsilon^{\alpha\beta\gamma}\Delta^{\mu\alpha}(x-w) \\ &\qquad \Delta^{\nu\beta}(y-w)\Delta^{\rho\gamma}(z-w) \end{aligned} \tag{4.5}$$

where,

$$\Delta^{\mu\nu}(x) = \epsilon^{\mu\nu\sigma}\frac{x_\sigma}{|x|^3} \tag{4.6}$$

Notice that in this expression there is a path integral and a space integral. Though its invariance under continuous deformations of the path T is a consequence of Chern-Simons gauge theory, it is worth proving that it is so. This has been in fact achieved.

Perturbation theory provides an infinite series of numerical invariants as the one shown. These invariants can be identified as Vassiliev invariants or numerical invariants of finite type. V. A. Vassiliev introduced his invariants in 1989 studying the cohomology of the space of all knots. These invariants have the property that if one defines from them invariants for singular knots using the equation:



there exists a finite value n such that for knots with $n+1$ singular points it vanishes. These values of n determine their orders or degrees. It turns out that the coefficient of the power series expansion of a Wilson loop which multiplies $1/k^n$ is a numerical knot invariant of order n. The invariant shown in (4.5) is of degree two.

Different representations of different groups provide different polynomial invariants and therefore different integral expressions for Vassiliev invariants. One could ask if at a given order in perturbation theory one could extract from the power series coefficient the contribution from the representation and group chosen. The answer to this question is positive due to the property of factorization intrinsic to the Feynman rules of Chern-Simons gauge theory. The power series expansion can be written as:

$$W_C = \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(C) r_{ij}(G, R) \frac{1}{k^i}. \tag{4.7}$$

The factor $r_{ij}(G, R)$ (group factor) contains all the dependence on the group and representation chosen, while the factor $\alpha_{ij}(C)$ (geometrical factor) contains all the dependence on the path C. The quantity d_i denotes the number of independent group factors. Let us explain what is meant by this. The Feynman rules generate many more group factors than d_i. However, if one considers their possible values in the space of all representations of all semi-simple groups one observes that all of them can be written in terms of just a few. A minimum set of these few is selected as the set of independent group factors. In fact, these factors build a vector space and what one is doing in (4.7) is just to choose a basis. Its dimension is the number d_i in (4.7). The dimensions d_i, which are known only up to order 9, are shown in Table 1. The geometrical factors $\alpha_{ij}(C)$

i	1	2	3	4	5	6	7	8	9
d_i	0	1	1	3	4	9	14	27	44

Table 1: Numbers of independent group factors

constitute a basis of Vassiliev invariants of order i.

As we have described, Chern-Simons gauge theory provides integral expressions for Vassiliev invariants. But the description presented is not the only one available to obtain representations of Vassiliev invariants from Chern-Simons gauge theory. There are many other ways to do perturbation theory, each providing a different representation. Chern-Simons gauge theory is a gauge theory and therefore has a gauge symmetry. Gauge invariant quantities like the Wilson loop can be computed in different gauges all leading to the same result. The expression presented in (4.5) is obtained in a specific gauge. Other gauge would lead to a different expression providing an alternative representation.

Given the space of all representations of all semi-simple Lie groups one obtains from Chern-Simons gauge theory an infinite sequence of sets of Vassiliev invariants. One could ask if the Vassiliev invariants so obtained are a complete set. In other words, that there is not a finite type invariant of a given order which cannot be expressed in terms of the ones originated from Chern-Simons gauge theory. The answer to this question seems to be negative. Possibly a structure bigger than semi-simple Lie groups is needed to accommodate all Vassiliev invariants.

Another important subject related to Chern-Simons gauge theory is the study of its partition function. This quantity leads to very interesting three-manifold invariants.

Chern-Simons gauge theory has opened a variety of new points of view in knot theory and on the topology of three-manifolds. Its field-theoretical study using non-perturbative and perturbative methods has provided a rich framework to analyze its topological invariants. A consequence of this analysis is that polynomial invariants based on representations of semi-simple Lie groups are in the same universality class of invariants as a subset of Vassiliev invariants. By being in the same universality class of invariants we mean that all the topological information which can also be obtained from one set of invariants can be obtained from the other. If two non-equivalent knots have the same polynomial invariant for all representations of all semi-simple Lie groups, they will clearly have the same Vassiliev invariants, at least of the mentioned subset. It is known that Chern-Simons gauge theory for semi-simple Lie groups do not detect non-invertible knots and mutant knots. The question of whether Vassiliev invariants ever discriminate among these types of knots is open.

5 Supersymmetry and Donaldson-Witten theory

In our previous attempt to find a non-abelian version of the Aharanov-Bohm effect we had to lower the dimension of space-time to construct a TQFT. The result was Chern-Simons gauge theory. There exist, however, TQFTs in four dimensions which are non-abelian gauge theories. In fact, there are two types: theories which are related to supersymmetry and theories which are not. The last set contains theories which share some common features with Chern-Simons gauge theory and are called BF theories. We will not discuss them here. Among the theories related to supersymmetry the first TQFT formulated by Witten in 1988 stands out. This theory deals with Donaldson invariants for oriented smooth four-manifolds. Theories in this second set present a series of special features which characterize them. But before going into detail let us make a short detour to introduce supersymmetry.

Supersymmetric quantum field theories possess a symmetry which consists of the invariance of the theory under a transformation which interchanges bosons and fermions. There are theories in which this interchanging can be done in different ways and then one has theories with $N = 2, 3, 4, \dots$ supersymmetries. For gauge theories in four dimensions $N = 4$ is the maximum number of supersymmetries if one excludes particles with spin two and higher. Particles in supersymmetric theories appear grouped into multiplets. For $N = 4$ there is only one multiplet, the gauge multiplet. For $N = 2$ there are two types of multiplets: gauge or vector multiplets, and matter multiplets or hypermultiplets.

Theories with a higher number of supersymmetries are simpler to solve but much more restricted. For example, $N = 4$ supersymmetric gauge theory does not renormalize and is conformal invariant. However, this theory is very restrictive and the only freedom is the choice of gauge group. $N = 4$ supersymmetric gauge theory has some common features with Chern-Simons gauge theory in three dimensions and one expects that soon it could be solved exactly. In fact, fundamental progress has been made in this direction during the last three years.

As it will be discussed below, $N = 4$ supersymmetric gauge theories possess a symmetry called duality which is extremely helpful towards the search of its exact solution. Though, strictly speaking, duality is not a symmetry for theories with a lower number of supersymmetries, it constitutes a helpful tool to analyze these theories and, indeed, its use has led to substantial progress towards the search for their exact solution. The use of the resulting information about this solution has led to the discovery of a new point of view in the theory of Donaldson invariants.

TQFTs of the type under consideration can be regarded as originated from supersymmetric theories with a least two supersymmetries. Theories with $N = 2$ supersymmetry are less restrictive than theories with $N = 4$ and can be labeled by a Lie group and a finite number of representations. Starting with an $N = 2$ supersymmetric theory one obtains a TQFT through the process of twisting. On flat space, a twisting consists of a rewriting of the theory in such a way that some fields are relabeled so that they have exotic new labels. Recall when we introduced quantum field theory that we attached labels to particles denoting their mass, their spin, etc. For the case of fields one also possesses a set of labels to characterize them. Of particular importance are the labels denoting their representation respect to the space-time group: the Lorentz group. In this sense one talks about scalar fields, spinor fields, vector fields, etc. Furthermore, $N = 2$ supersymmetric theories have an extra symmetry together with the space-time symmetry. This symmetry is called internal and its group is $SU(2)$. Fields also carry labels indicating how they transform under the internal symmetry group. A twisting consists of choosing an exotic relabeling of the fields or a particular mixing between the space-time symmetry group and the internal symmetry group. For theories with only two supersymmetries this can be done only in one non-trivial way while for theories with $N = 4$ there are three non-equivalent ways.

The theory resulting after the twisting is the same as the original one on flat space-time. However, it is different when considered on curved space. The reason is that the coupling of the fields to the background Riemannian metric is dictated by their spin, which has been changed in the twisting. Twisted theories have three important properties. First, they have a scalar symmetry even when they are considered on an arbitrary smooth four-manifold. Second, due to the presence of this symmetry these theories are such that the vacuum expectation values of quantities which are invariant under this symmetry are invariant under deformations of the Riemannian metric. Third, again due to the presence of the scalar symmetry, the vacuum expectation values are independent of the coupling constant of the theory. This is not exactly true for twisted theories originated from $N = 4$ supersymmetric gauge theories where there remains a dependence which, however, is simple to control. We will exclude these theories in our discussion. These properties indicate that vacuum expectation values are just numbers (not functions of the coupling constant as in Chern-Simons gauge theory) which are topological invariants of the four-manifold where the theory is defined. The input data which characterize these vacuum expectation values are labeled by the homology of the four-manifold. To a specific selection of homology cycles correspond a number which is a topological invariant.

The topological invariants which are obtained after the twisting of an $N = 2$ supersymmetric gauge theory with no representation labels and gauge group $SU(2)$ are Donaldson invariants. This was shown by Witten in his seminal paper of 1988. He proved this connection using perturbative methods. The basic idea is the following. Twisted theories are TQFTs whose vacuum expectation values are independent of the coupling constant of the theory. This means that the calculation of these quantities in the $g \to 0$ limit is exact. But the $g \to 0$ limit is rather simple: one has just to keep the first term of the perturbative series expansion. This was done by Witten in 1988 showing that the resulting expression were the same as the ones proposed by Donaldson to define his invariants for four-manifolds. This was rather satisfactory because, finally, Atiyah's proposal of giving a quantum field theory interpretation to Donaldson theory was implemented. However, Witten's formulation did not lead to further progress towards the computation of these invariants.

Let us briefly discuss what kind of invariants one is dealing with in Donaldson-Witten theory. The perturbative analysis of the theory leads to the conclusion that one has to compute certain quantities on the space of solutions of a set of equations which are very familiar in physics, the instanton equations,

$$F^{+}_{\mu\nu} = 0, \tag{5.1}$$

where $F_{\mu\nu}$ is the field strength or curvature associated to the gauge connection A_μ, and the symbol plus indicates that one is equating to zero only the self-dual part. The solutions of this equation are called instantons and the space formed by those solutions is the moduli space of instantons, which will be denoted by $\mathcal{M}$. In this space two instanton solutions which are related by a gauge transformation are considered equivalent. From the input data, which, as indicated, were labeled by homology cycles, $\gamma_1, \gamma_2, \ldots$, the perturbative analysis leads to a well defined prescription to map to each set of labels a cohomology cocycle $\Omega_{\gamma_1,\gamma_2,\ldots}$ on the moduli space of instantons $\mathcal{M}$. The integrals of these forms over the moduli space,

$$\int_{\mathcal{M}} \Omega_{\gamma_1,\gamma_2,\ldots}, \tag{5.2}$$

are the numbers which correspond to Donaldson invariants. The problem related to the compactification of this moduli space (in general it is not compact) is the same one as in Donaldson theory. From the perturbative point of view TQFT does not bring anything new to this problem, it just shows that the theory we are dealing with is in fact the TQFT of Donaldson invariants. Insight from quantum field theory could come if one were able to carry out the analysis of the theory for a different value of the coupling constant g, for example, $g \to \infty$, or strong coupling limit. However, the corresponding analysis required non-perturbative information which was not available until recently. Before getting into the non-perturbative analysis we need to discuss some aspects of duality.

6 Duality and Seiberg-Witten invariants

Electromagnetic duality is a symmetry of Maxwell's equations without matter which allows to interchange the electric and magnetic fields. If one writes Maxwell's equations in terms of the complex field $\mathbf{E} + i\mathbf{B}$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields respectively,

$$\begin{aligned} \nabla \cdot (\mathbf{E} + i\mathbf{B}) &= 0, \\ \nabla \wedge (\mathbf{E} + i\mathbf{B}) &= i\frac{\partial}{\partial t}(\mathbf{E} + i\mathbf{B}), \end{aligned} \tag{6.1}$$

duality is the invariance of these equations under the transformation:

$$\mathbf{E} + i\mathbf{B} \to e^{i\phi}(\mathbf{E} + i\mathbf{B}). \tag{6.2}$$

When matter is included in Maxwell's equation, duality is only maintained if one assumes that matter is composed of classical point particles carrying electric and magnetic charges. If these charges are q and g respectively, duality is kept if these transform as:

$$q + ig \to e^{i\phi}(q + ig). \tag{6.3}$$

The price one has to pay to preserve duality is the inclusion of unobserved magnetic charge.

As was discussed before, the quantum description of the coupling of charged particles to electromagnetic fields is made using the electromagnetic potential. In the presence of magnetic charges the coupling is consistent only if some constraints are satisfied. In 1931 Dirac proved that a magnetic charge g_1 carrying no electric charge could occur in the presence of an electric charge q_2 carrying no magnetic charge provided the following condition is satisfied:

$$q_2 g_1 = 2\pi n\hbar, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.4}$$

being $\hbar$ the Plank's constant. This is known as the Dirac quantization condition and it implies that if a magnetic charge g_1 exists, electric charge is quantized. Quantization of electric charge is a feature of nature and this explanation is perhaps the best yet found. Particles carrying only magnetic charge are called monopoles.

One of the problems with Dirac's quantization condition is that it is not invariant under duality. It took some time to realize how this condition has to be generalized to accommodate duality. The new input is to assume that there are particles carrying electric and magnetic charges. These particles are called dyons. Applying Dirac's argument to dyons carrying, respectively, charges (q_1, g_1) and (q_2, g_2) one finds:

$$q_1 g_2 - q_2 g_1 = 2\pi n\hbar, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.5}$$

This is known as Schwinger quantization condition and it is invariant under the duality transformation (6.3). One of the consequences of Schwinger quantization condition is that the set of possible electric and magnetic charges form a two-dimensional lattice. This is a property that must be satisfied by the electric and magnetic charges of the particle spectrum of any quantum field theory having duality as a symmetry.

During the last years, evidence has accumulated to make plausible that $N = 4$ supersymmetric $SU(2)$ gauge theory is a theory where duality is realized exactly. In this theory there is a part of the spectra obtained by spontaneous symmetry breaking. The rest of the spectra is realized through monopole and dyon solitons. A consequence of duality is that this theory possesses many equivalent descriptions. For example, one could choose to describe it via a Higgs mechanism applied to some other part of the spectra, realizing now the original ones as solitons. This indeed can be done provided one changes properly the coupling constant of the theory. To choose a particular description is basically to make a choice of basis in the lattice of allowed electric and magnetic charges. Depending on the choice one has a different coupling constant. All these choices are related by a duality group of transformations. For example, there exist dual descriptions in which the coupling constant g is interchanged by $1/g$, in other words, the interchange of weak and strong couplings.

Although there is not a proof yet that in $N = 4$ supersymmetric $SU(2)$ gauge theory duality is exactly realized, this has been verified in one of its twisted versions. The partition function of this twisted theory has been computed for some four-manifolds obtaining a result which is invariant under the full duality group. This question should be addressed for other gauge groups and for the other two non-equivalent twistings of $N = 4$ supersymmetric gauge theory.

$N = 2$ supersymmetric gauge theories are rather different than their $N = 4$ counterpart. The first important difference is that in general these theories are not conformal invariant and therefore the coupling constant gets renormalized. The second difference is that in $N = 2$ supersymmetry there exist two kinds of supersymmetric multiplets, the gauge multiplet and the matter multiplet or hypermultiplet. In this theories one does not expect duality to be realized exactly. However, there is a variant of this symmetry which plays a fundamental role.

$N = 2$ supersymmetric gauge theory is asymptotically free. This means that at high energy (ultraviolet regime) the theory is weakly coupled, the effective coupling constant becomes small. At low energies (infrared regime) the theory is strongly coupled becoming its effective coupling constant big. Seiberg and Witten discovered that for $N = 2$ supersymmetric gauge theories duality become the statement that the strongly coupled limit is equivalent to the weak coupling limit of some other system. They found that system for the case under consideration. Notice that the statement is consistent with what we found for $N = 4$ supersymmetric gauge theories. What distinguishes $N = 4$ is that the 'other system' is again $N = 4$ supersymmetric $SU(2)$ gauge theory. In $N = 4$ supersymmetry there is only one multiplet and therefore the 'other system' has to be of the same type. Only the gauge group could be modified. In fact, that seems to be the case when considering more complicated groups. In $N = 2$ supersymmetry there are two multiplets and therefore there are many more possibilities for the 'other system'. Seiberg and Witten found that the strongly coupled limit of $N = 2$ supersymmetric $SU(2)$ gauge theory is equivalent to a weakly coupled $N = 2$

supersymmetric abelian gauge theory coupled to matter hypermultiplets.

In the weak coupling limit, or perturbative regime, one deals with the space of classical vacua of the theory. For $N = 2$ supersymmetric $SU(2)$ gauge theory this space is parametrized by a complex parameter u. Of particular importance, specially in its application to TQFT, is the massless spectra for each value of u. It turns out that for $u \neq 0$, since the gauge symmetry is spontaneously broken, there is only one massless particle: a photon described by an abelian gauge field. At $u = 0$ the full gauge symmetry is restored and there are three massless particles corresponding to the three gauge bosons. This point is called singular.

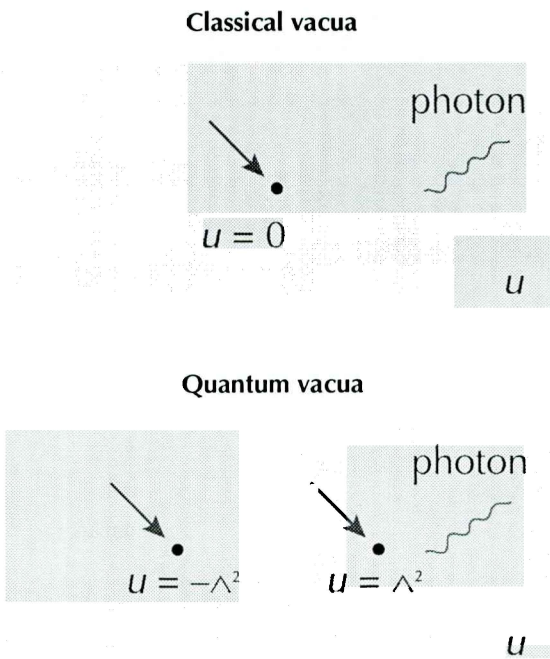


Figure 4: Classical and quantum vacua.

In an asymptotically free theory, as the one under consideration, the strong coupling limit correspond to the quantum vacua of the theory. Seiberg and Witten showed that this space of vacua is parametrized again by a complex parameter u. For values of $u \neq \pm\Lambda^2$, where Λ is certain mass scale, the only massless particle corresponds to an abelian gauge field. For $u = \pm\Lambda^2$ there are additional massless particles: among them a magnetic monopole for $u = \Lambda^2$ and a dyon for $u = -\Lambda^2$. Full $SU(2)$ symmetry is never restored. At the quantum vacuum $u = \Lambda^2$ the weakly coupled theory is an $N = 2$ supersymmetric abelian gauge multiplet coupled to a massless hypermultiplet. The points $u = \pm\Lambda^2$ are called singular points. The classical and quantum moduli spaces of vacua are represented in Figure 4.

In the previous section we carried out the perturbative analysis of Donaldson-Witten theory. This analysis was done for $g \to 0$ and therefore it corresponds to the ultraviolet regime or weakly coupled limit. Since this TQFT is independent of g the exact result is just a sum over classical vacua. We actually did not do this in our analysis of the previous section. There, of all the values of u we just took the contribution from $u = 0$. We ignored the rest. This is justified if the manifold satisfies the topological property $b_2^+ > 1$. We succinctly assumed this to hold in the previous perturbative analysis. The condition $b_2^+ > 1$ means that the number of self-dual cohomology 2-cocycles is bigger than 1. Precisely smooth manifolds satisfying this condition are the most studied in Donaldson theory. The condition $b_2^+ > 1$ implies that the contributions from $u \neq 0$ vanish.

The analysis of Donaldson-Witten theory in the strong coupling limit should provide a new point of view on Donaldson invariants. This seemed hard to achieve before 1994 but after Seiberg and Witten's work on the strong coupling limit of $N = 2$ supersymmetric $SU(2)$ gauge theory, this goal was reached. The main, piece of the argument is that in the strong coupling limit the contributions come only from the space of quantum vacua. Again, the condition $b_2^+ > 1$ notably simplifies the analysis because in this case only contributions from the points $u = \pm\Lambda^2$ survive. Actually, it is enough to work out the contribution from $u = \Lambda^2$ since there is a symmetry which relates both points. At $u = \Lambda^2$ the weakly coupled theory is known and one has just to work out its twist. The result is obtained using the previous perturbative methods in this weakly coupled theory and one finds that the contributions come from the solutions of a different set of equations:

$$\begin{aligned} &F^+_{\mu\nu} + \overline{M}\gamma^+_{\mu\nu}M = 0, \\ &\gamma^\mu \nabla_\mu M = 0. \end{aligned} \tag{6.6}$$

In these equations M is a commuting chiral spinor, and γ_μ, $\gamma^+_{\mu\nu}$ are Dirac matrices. These equations are known as the monopole equations or Seiberg-Witten equations. They are simpler than the instanton equations because the field strength $F_{\mu\nu}$ corresponds to an abelian gauge field. The second equation in (6.6) is just the Dirac equation for the chiral spinor M. The vacuum expectation values analyzed in the perturbative approach can be rewritten now as a sum over solutions of the Seiberg-Witten equations. Actually, these sums have a very simple form. They turn out to be:

$$\sum_{x \in \Gamma} n_x f_{\gamma_1, \gamma_2, \ldots}(x) \tag{6.7}$$

where the n_x are the Seiberg-Witten invariants. In this equation Γ is a set of cohomology classes x which satisfy certain constraint (also known as basic classes) and $f_{\gamma_1, \gamma_2, \ldots}(x)$ is a function of x which involves the input data. Recall that this input data consisted of a finite set of homology cycles $\gamma_1, \gamma_2, \ldots$. The Seiberg-Witten invariants n_x involve a sum over solutions of the Seiberg-Witten equations for an abelian gauge field in the class x. In fact, n_x is just the partition function of a TQFT for fixed x.

Equation (6.7) presents some similarities with the one found for Chern-Simons gauge theory involving Vassiliev invariants. There, the Vassiliev invariants contained all the topological information on the knot. Here, the Seiberg Witten invariants contain all the topological information on the the smooth four-manifold.

Seiberg-Witten invariants were totally unexpected in mathematics. They certainly have opened a new door. Conjectures about four-manifolds which were waiting for a proof based on Donaldson theory were quickly proved using Seiberg-Witten invariants. At the moment we are lacking a proof of equation (6.7), but this is very hard with today's knowledge, and perhaps not the most interesting thing to do. Seiberg-Witten invariants stand out by themselves and can be used disregarding their origin from Donaldson theory. It is very likely that the first proof of (6.7) will come from string theory. Nevertheless, further developments of this theory are necessary before having a glimpse on how this could be achieved.

7 Non-abelian monopoles

We have limited our discussion to Donaldson-Witten theory with gauge group $SU(2)$. Certainly, we could ask about why not to consider other groups and couplings to twsited $N=2$ hypermultiplets. TQFTs of these types are in general labeled by a group and a finite number of representations which denote the ones chosen for the twisted hypermultiplets. No much has been explored in this direction. Only the case of $SU(2)$ with a hypermultiplet in its fundamental representation has been studied. We will briefly describe it in what follows.

The perturbative analysis is similar to the one in Donaldson-Witten theory. The input data are labeled in the same way and one ends with an integration over the moduli space of the non-abelian version of the monopole equations (6.6):

$$\begin{aligned} & F^{a+}_{\mu\nu} + \overline{M}\gamma^{+}_{\mu\nu}T^{a}M = 0, \\ & \gamma^{\mu}\nabla_{\mu}M = 0, \end{aligned} \tag{7.1}$$

where T^a is an $SU(2)$ generator. These equations are called non-abelian monopole equations. The resulting moduli space contains the moduli space of instantons as a subset. It has basically the same type of problems as that moduli space. The non-perturbative analysis of the physical theory has been done by Seiberg and Witten. As in the previous case, the quantum vacua possess a massless abelian gauge field, but now it contains three singular points related by a symmetry. One of these points corresponds again to a massless magnetic monopole and the contributions come again from solutions of the abelian monopole equations (6.6). The final expression in the strong coupling limit can be written as:

$$\sum_{x\in\Gamma} n_x \tilde{f}_{\gamma_1,\gamma_2,\ldots}(x) \tag{7.2}$$

where n_x are the same Seiberg-Witten invariants as before. However, the function multiplying them, $\tilde{f}_{\gamma_1,\gamma_2,\ldots}(x)$, is different. Comparing the results (6.7) and (7.2) we can assure that no new topological information is obtained analyzing the moduli space of non-abelian monopoles. All that information is already contained in the moduli space of instantons.

These observations bring again the idea of universality classes of topological invariants. It seems that Seiberg-Witten invariants represent a class in the sense that topological invariants associated to several moduli spaces can be written in terms of them. This is certainly true for the two cases studied but presumably it holds for other groups. It is very likely that Seiberg-Witten invariants are the first set of a series of invariants, each defining a universality class. TQFTs originated from the twist of $N = 2$ supersymmetric gauge theory constitute a big set of theories labeled by the group and a finite number of representations. Only two elements of this set have been studied. Presumably, many more new invariants and many more new relations among invariants of different moduli spaces are waiting there to be discovered.

8 Final remarks

In this talk I have described several examples which show how ideas from quantum field theory and, in particular, from TQFT have been very successful in the discovery of new results in the topology and geometry of smooth low-dimensional manifolds. We have analyzed situations in which the physical approach occurred first leading to new mathematics, and situations in which, though mathematics came first, physics provided an important generalization. The examples described, and many other which we have not treated, show that TQFT makes correct predictions in mathematics. These days, quantum field theorists, though working with a tool which is not rigorous, are being encouraged not only by the excellent experimental agreement achieved by physical theories, but also by the success accumulated with this other type of predictions. Physicists and mathematicians should join efforts to construct a rigorous and sound foundation for quantum field theory.

TQFTs are simpler than ordinary quantum field theories and presumably it is easier to make them rigorous. The difficulties in their rigorous definition by analytic methods could be overcome by axiomatizing them. In fact some TQFTs can be constructed using combinatorial and algebraic methods. However, it is likely that the richness inherent to the methods developed by quantum field theorists is a much more powerful tool to obtain unexpected relations between different sets of invariants, or a variety of representations for each of them. In this talk the success of the these methods has been described for theories in three and four dimensions. The results are summarized in Table 2, where the invariants and the methods used in their analysis are presented.

	$d=3$	$d=4$
perturbative	Vassiliev	Donaldson
non-perturbative	Jones	Seiberg-Witten

Table 2: Topological invariants in the perturbative and the non-perturbative regimes for $d=3$ and $d=4$.

Physicists have started to accumulate a big amount of knowledge on the behavior of $N=2$ supersymmetric gauge theories. The application of these results to TQFT has led to the prediction of Seiberg-Witten invariants. This should be regarded as a first result of possibly a series of important relations between different sets of topological invariants. Duality would be at the heart of these developments. There is some evidence that duality has its roots in string theory and that the evolution of this theory will provide new insights in supersymmetric physical theories and in their topological counterparts. From this point of view duality might relate also different sets of invariants for manifolds with dimension different than four. Some results in this direction have been recently obtained in three dimensions. String theory itself could provide new unexpected results in geometry and topology. However, though a considerable amount of progress has been made in the last years, we are still far from the fundamental formulation of string theory. What is becoming firmly accepted is that in such a formulation duality will play an important role. This is a very encouraging feature towards future developments of TQFT. The best is yet to come.

Acknowledgements

I would like to thank D. Jou and the organizers of the "Jornades sobre Física i Geometria" for inviting me to give this talk. I would like to thank also A.V. Ramallo and my collaborators M. Alvarez, C. Lozano, M. Mariño and E. Pérez for very helpful discussions on many aspects of the subjects treated in this talk.

References

[1] V.F.R. Jones, *Bull. AMS* **12** (1985) 103; *Ann. of Math.* **126** (1987) 335.

[2] M. F. Atiyah, "New invariants of three and four dimensional manifolds", in *The Mathematical Heritage of Herman Weyl, Proc. Symp. Pure Math.* **48**, *American Math. Soc.* (1988) 285-299.

[3] E. Witten, *Comm. Math. Phys.* **117** (1988) 353.

[4] E. Witten, *Comm. Math. Phys.* **121** (1989) 351.

[5] S. K. Donaldson, *Topology* **29** (1990) 257.

[6] V.A. Vassiliev, "Cohomology of knot spaces" in *Theory of Singularities and its applications*, (V.I. Arnold, ed.), *Amer. Math. Soc.*, Providence, RI, 1990, 23.

[7] N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994) 19; Erratum, *Nucl. Phys.* **B430** (1994) 485; *Nucl. Phys.* **B431** (1994) 484.

[8] E. Witten, *Math. Res. Lett.* **1** (1994) 769.